31765: Optimization in modern power systems

Lecture 8: Duality

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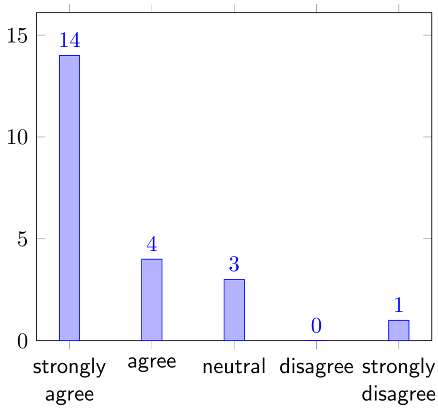
The Goals for Today!

- Review of Mid-Term Evaluation
- Duality
- Duality in LP

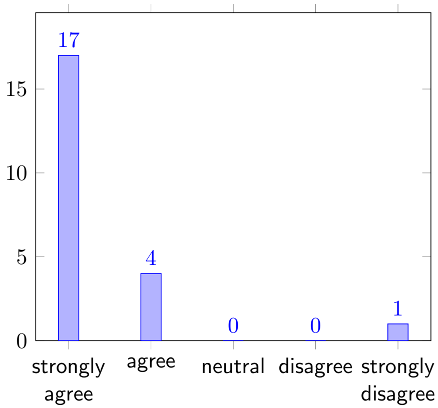
Review of Mid-Term Evaluation

- 22 questionnaires (out of 25 participants)

I think I am learning a lot in this course

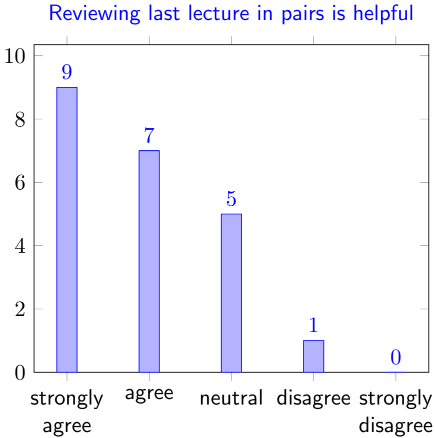


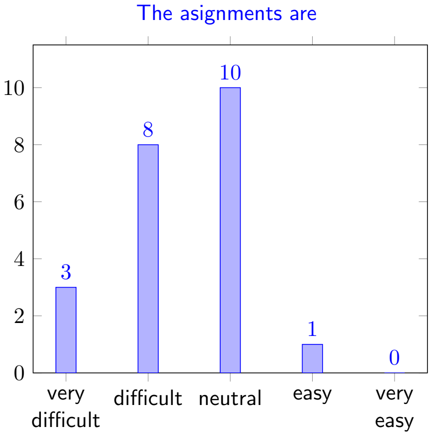
In general, I think this is a good course



- 21 out of 22 find the course good! (95%) ☺
- 18 out of 22 think they learn a lot (82%)

Review of Mid-Term Evaluation





- 73% find the 5-min reviewing helpful
- The assignments are considered average or difficult

Review of Mid-Term Evaluation

- Good:
  - Teacher and Teaching
  - Time for questions
  - real-life relevant course
- Bad:
  - Not enough time for assignments and studying
  - 2-hour lectures are not enough
- Comments to improve the course:
  - Matlab guidance! (several comments)
  - Spend more time in class (all 4 hours); longer lectures
  - 8am is too early
  - textbook

Dual Problem

With the help of the Lagrangian function and the Lagrangian multipliers, we can define and solve a dual optimization problem.

- Primal problem: our original problem
- Dual problem: the problem we formulate with the help of the Lagrangian
- Dual variables $\equiv$ Lagrangian multipliers

Why do we care about the dual?

Advantages of the dual problem:

- it might be easier to solve, e.g. less constraints
- always concave $\rightarrow$ convex optimization
- always gives a lower bound to the objective value of our original problem
- for certain set of problems, e.g. convex $\rightarrow$ exact
  - Strong duality $\rightarrow$ The dual problem of convex primal problems *usually* results to the same solution as the primal problem

The dual function is concave

$$g(\lambda,\nu) = \inf_{x\in D} L(x,\lambda,\nu) = \inf_{x\in D}\left(f_0(x) + \sum_i \lambda_i f_i(x) + \sum_i \nu_i h_i(x) \right)$$

- $\inf_{x\in D}$ stands for the minimum value of the Lagrangian over x: for $\lambda \in R^m, \nu \in R^p$
- g is always concave: Lagrangian is linear with respect to λ, ν and $\inf$ preserves concavity
- The dual function is concave, even if f_0, f_i, h_i are non-convex/non-concave.

The dual function is concave: Example

$$\min x_1^2 + x_2^2$$

subject to:

$$x_1 + x_2 - 4 = 0$$

Find the dual function $g(\nu) = \inf_{x \in D} L(x, \nu)$

The dual function is concave: Example

$$\min x_1^2 + x_2^2$$

subject to:

$$x_1 + x_2 - 4 = 0$$

Find the dual function $g(\nu) = \inf_{x \in D} L(x, \nu)$

$$L(x,\nu) = x_1^2 + x_2^2 + \nu(x_1 + x_2 - 4)$$
$$g(\nu) = \inf_{x \in D} L(x,\nu) \Rightarrow \nabla_x L = 0$$

$$\nabla_x L = \begin{bmatrix} \frac{\partial L}{\partial x_1} \\ \frac{\partial L}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 + \nu \\ 2x_2 + \nu \end{bmatrix} = 0 \Rightarrow \begin{matrix} x_1 = -\frac{\nu}{2} \\ x_2 = -\frac{\nu}{2} \end{matrix}$$

$$L(\nu) = -\frac{\nu^2}{2} - 4\nu \Rightarrow \text{concave!}$$

Dual function: lower bound

- For *any* $\lambda \geq 0$ and *any* ν, it holds:

$$g(\lambda, \nu) \leq f_0(x^*)$$

- Assume $\tilde{x}$ feasible point, i.e. $f_i(\tilde{x}) \leq 0,\ h_i(\tilde{x}) = 0,\ \lambda \geq 0$. Then we have

$$\sum_i \lambda_i f_i(\tilde{x}) + \sum_i \nu_i h_i(\tilde{x}) \leq 0$$
$$L(\tilde{x}, \lambda, \nu) = f_0(\tilde{x}) + \sum_i \lambda_i f_i(\tilde{x}) + \sum_i \nu_i h_i(\tilde{x}) \leq f_0(\tilde{x})$$
$$g(\lambda, \nu) = \inf_{x \in D} L(\tilde{x}, \lambda, \nu) \leq L(\tilde{x}, \lambda, \nu) \leq f_0(\tilde{x})$$

- This holds for every feasible point $\tilde{x}$, including the optimal point x^*.

Strong and weak duality

- Dual problem:

$$\max g(\lambda, \nu)$$
$$\text{subject to } \lambda \geq 0$$

- Always a convex problem!
- Weak duality: $g(\lambda^*, \nu^*) \leq f_0(x^*)$
- Strong duality: $g(\lambda^*, \nu^*) = f_0(x^*)$
- Duality gap: $g(\lambda^*, \nu^*) - f_0(x^*)$
- Strong duality usually holds for convex problems!

- Dual: convex & lower bound $\Rightarrow$ Cheap certificate!
- If $g(\lambda^*, \nu^*) = f_0(x^*)$, it's guaranteed that this is the global optimum

Strong duality: example

$$\min x_1^2 + x_2^2$$

subject to:

$$x_1 + x_2 - 4 = 0$$

Dual:

$$L(\nu) = -\frac{\nu^2}{2} - 4\nu$$

1. Find $\min_x f_0(x)$ s.t. $h(x) = 0$
2. Find $\max_\nu L(\nu)$

- What do you observe?
- Which problem is it easier to solve?

Dual of a Linear Program

LP in standard form

$$\min c^T x$$
$$\text{subject to } Ax = b$$
$$x \geq 0$$

$\Leftrightarrow$

Dual Problem

$$\max \; -b^T \nu$$
$$\text{subject to } A^T \nu + c \geq 0$$

minimize
$\#n$ variables x
$\#p$ equality constraints

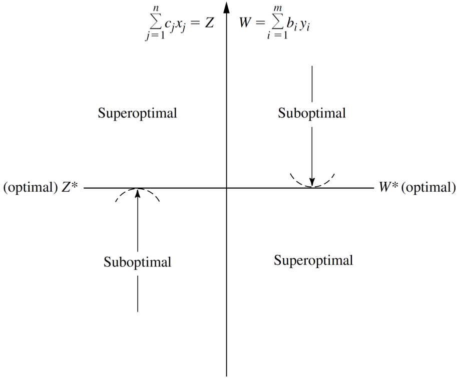
$\Leftrightarrow$

maximize
$\#n$ inequality constraints
$\#p$ dual variables ν

- $-\nu^T b = -\nu^T Ax \leq c^T x$: if x and ν are feasible solutions, $-b^T \nu \leq c^T x$.
- if x^* and ν^* are feasible solutions and $-b^T \nu = c^T x$, then x^* and ν^* are the optimal solutions for their respective problems.

Two different paths with the same endpoint

Dual problem   Primal Problem



Slide inspired from Juan-Miguel Morales, 02435 Decision-Making under uncertainty in Electricity Markets, DTU.

Figure taken from: F.S. Hillier, G.J. Liebermann. Introduction to Operations Research. McGraw Hill, 2001.

Strong duality and KKT conditions

$$L(x,\lambda,\nu) = f_0(x) + \sum_i \lambda_i f_i(x) + \sum_i \nu_i h_i(x) \qquad (1)$$

Strong duality: When does $L(x^*,\lambda,\nu) = f_0(x^*)$ hold?

Remember:

$h_i(x) = 0$

$f_i(x) \leq 0$

$\lambda_i \geq 0$

Strong duality and KKT conditions

$$L(x, \lambda, \nu) = f_0(x) + \sum_i \lambda_i f_i(x) + \sum_i \nu_i h_i(x) \qquad (1)$$

Strong duality: When does $L(x^*, \lambda, \nu) = f_0(x^*)$ hold?

Remember:

$h_i(x) = 0$

$f_i(x) \leq 0$

$\lambda_i \geq 0$

When $\lambda_i f_i(x^*) = 0$

(complementary slackness)

KKT conditions hold only if strong duality exists

- KKT conditions require that $\lambda_i f_i(x^*) = 0$

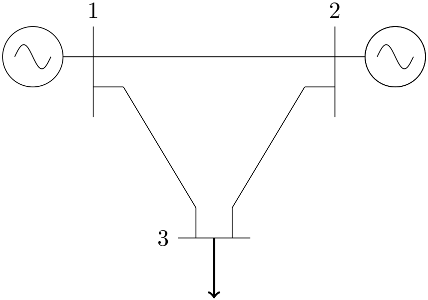
KKT conditions hold only in case of strong duality

- Strong duality *usually* holds in convex problems $\Rightarrow$ DC-OPF is convex
- Convex problems with strong duality: KKTs are necessary and sufficient.

Convex problems (such as DC-OPF):
If any point satisfies the KKT conditions, then it is the global optimal.

- We can solve either the primal or the dual problem: same objective value at x^*, due to strong duality

Question: What is the dual of the DC-OPF?



$$\min c_1 P_{G1} + c_2 P_{G2}$$

subject to:

$$B\theta = P_G - P_L$$

$$P_G \geq 0$$

- no line flow constraints

Duality: Wrap-up

- The dual problem is a convex optimization problem
- Lower bound and weak duality: if x^* and λ^*, ν^* feasible, then $g(\lambda^*, \nu^*) \leq f_0(x^*)$
- Strong duality: if x^* and λ^*, ν^* feasible solutions and $g(\lambda^*, \nu^*) = f_0(x^*)$, then x^* and λ^*, ν^* are the optimal solutions for their respective problems.
- If dual unbounded above, the primal is infeasible – and vice versa: if primal unbounded below, the dual is infeasible.
- The dual can provide a cheap certificate for a lower bound of the objective value.
- In general if the primal has more constraints than variables, the dual will have more variables than constraints:
  - less constraints $\rightarrow$ easier to solve

Special Course on Large-Scale Optimization and Decomposition

- How do you design an OPF for a very large power grid, with thousands of distributed generators, batteries, and uncertain variables?
- Large-scale optimization problems can be very difficult to solve
- Decomposition methods help break one big problem into smaller problems
- Lecturer: Asst. Prof. Jalal Kazempour
- When: Jan 2 – Jan 19, 2018, 5 ECTS
- See also course description in our Campusnet